THE PRUNOID OR PLUM CURVE.

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In a former number of the ANALYST (p. 111, Vol. VI) I had occasion to discuss the Polar Equation,

$$\rho = \frac{m + \cos\theta}{m^2 - 1}.$$

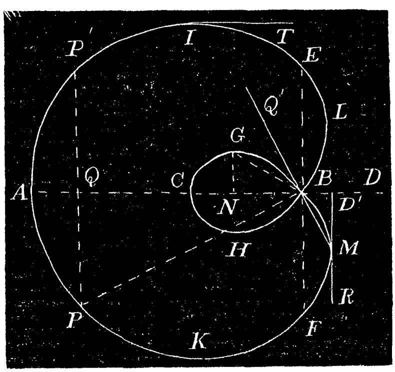
Making $m > 1$, we found as a result what was called the Curve of Capture. At the time, I did not intend to refer to this equation again, but some results of further consideration seem to me worthy the attention of mathematicians.

I. Let $m = 1$; then $\rho = \infty$ for every value of θ except 180°, and we have a circle of infinite radius for all values of θ other than 180°. For $\theta = 180°$ $\rho = 0 \div 0$, i. e., is indeterminate. The true value, however, is easily obtained:

$$\rho = \frac{m - 1}{m^2 - 1} = \frac{1}{m + 1} = \frac{1}{2}.$$ This gives an isolated point.

II. A much more remarkable result emerges by taking $m < 1$. In drawing the curve it is necessary to observe the signs closely. For a particular case, take $m = \frac{1}{2}$. Let $\theta = 0$; then is

$$\rho = \frac{m + \cos\theta}{m^2 - 1} = \frac{1.5}{-.75} = -2.$$



Lay off $BA = 2$ in. to the left of B.

Let $\theta = 90°$; $\therefore$ $\rho = BF = -\frac{2}{3}$.
" $\theta = 180°$; " $\rho = BC = \frac{2}{3}$.
" $\theta = 270°$; " $\rho = BE = -\frac{2}{3}$.
" $\theta = 360°$: " $\rho = BA = -2$.

For all values of θ between 0° and 90° we get points of the curve between A and F; and for all values between 270° and 360°, we get points between E and A. Let us now determine the points B and G on $FBGC$.

At B, $\rho = \dfrac{m + \cos\theta}{m^2 - 1} = 0$; $\therefore$ $m = .5 = -\cos\theta$; $\therefore$ $\theta = 120°$. Designating by G the highest point between B and C, let fall GN perpendicular to AB. GN is a maximum, $GB = \rho$. $GN = \rho \sin \angle GBN = \rho \sin\theta$. Generalizing we get $\dfrac{m + \cos\theta}{m^2 - 1}\sin\theta$, for the distance of the highest or lowest

point of the curve from the line AD; $\therefore$ $m\sin\theta+\cos\theta\sin\theta=$ a maximum or minimum, $\therefore$ $m\cos\theta+\cos^2\theta-\sin^2\theta=0$. Therefore, for $m=.5$,

$$\cos^2\theta+.25\cos\theta=.5,$$
$$\therefore \cos\theta=.5931 \text{ or } -.8931;$$
$$\therefore \quad \theta=53°37' \text{ or } 147°28'.$$

The former value belongs to K; the latter to G. It is evident that the curve is symmetrical in reference to the initial line AB; for, by using negative angles, $-\theta$ for $+\theta$, we construct the curve from A through I, E, L, B and H to C, and $\cos(-\theta)=\cos\theta$, always. Hence I and H are symmetrical with K and G.

To find the greatest and least values of $\rho=\dfrac{m+\cos\theta}{m^2-1}$. Omit the denominator, and $m+\cos\theta$ is greatest when $\theta=0$, viz., $.5+1=1.5$; it is least algebraically when $\theta=180°$, viz., $.5-1=-.5$, and it is least numerically when $\theta=120°$ or $240°$, viz., $.5-.5=0$. The successive points are A, C and the double point B. The corresponding values of ρ are -2, $+\frac{2}{3}$ and 0.

To find the angles at which the curve or its tangent cuts the initial line AD. The trigonometrical tangent of the angle formed by a tangent line to the curve and the radius vector to the point of contact is

$$\frac{\rho d\theta}{d\rho}=\frac{m+\cos\theta}{-\sin\theta}.$$

This becomes infinite only when $\sin\theta=0$; $\therefore$ for $\theta=0°$ or $180°$, the tangent $=\infty$; and therefore the curve is perpendicular to AD at C and A, and no where else. Also, the tangent to the curve makes a right angle with the radius vector at no other points.

To find the point where the angle between the tangent line and the radius vector is zero. Let $\dfrac{m+\cos\theta}{-\sin\theta}=0$; $\therefore$ $\cos\theta=-m=-.5$, $\therefore$ $\theta=120°$ or $240°$. This is at B.

Let the radius vector start from the position BA. The tangent line at A is at right angles with the r. v. As the r. v. revolves through K, F, M, to B, the angle increases from 90° to 180°. Setting out anew at B and proceeding by way of G to C, the angle may be conceived as beginning with 0° and reaching 90° again at C. It then increases up to 180° or coincidence again at the multiple point B; and once more setting out as from zero, goes back gradually to 90° at A.

While there are only two points, A and C, at which the tangent line is perp. to the r. v., it is manifest that there are two other points, as L and M, the tangent line through which is perpendicular to AD.

To find the point M. $DBQ' = \theta$. BMR is the angle between the tang. line and the radius vector. RM prolonged to D' is perpendicular to BD.

Tang. $BMR = -$ tang. $BMD' = -\dfrac{1}{\tan MBD} = \dfrac{1}{\tan\theta}$. But we have already seen that

$$\tan BMR = -\frac{m+\cos\theta}{\sin\theta};\ \therefore\ \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = -\frac{m+\cos\theta}{\sin\theta}.$$

$$\therefore\ 2\cos\theta = -m = -.5,\ \therefore\ \cos\theta = -.25;\ \therefore\ \theta = 104°29'.$$

$$BM = \rho = \frac{m+\cos\theta}{m^2-1} = \frac{.5-.25}{-.75} = -\frac{1}{3}.$$

Tests.—As it is always desirable to test our work, we notice that at the points I, G, H and K, the tangent line is parallel to the initial line AB. For example, let TI be the tang. line at I, and be parallel to BD. Then

$$\tan TIB = -\tan IBD = -\tan\theta;$$

$$\therefore\ -\frac{m+\cos\theta}{\sin\theta} = -\frac{\sin\theta}{\cos\theta};\ \therefore\ \cos^2\theta + m\cos\theta = \sin^2\theta.$$

This is the previous result over again.

There is a neat test also for the points A, C, L and M. Let us suppose it undetermined whether the farthest point to the left of B is on the initial line. It might be, say, at W. Draw WV perpendicular to BA at V.

$$BV = \rho\cos\theta = \frac{m+\cos\theta}{m^2-1}\cos\theta;\ \therefore\ m\cos\theta + \cos^2\theta = \text{a maximum.}$$

$$\therefore\ -m\sin\theta - 2\cos\theta\sin\theta = 0.$$

This equation may be verified either by $\sin\theta = 0$, and therefore $\theta = 0°$ or $180°$, giving the points A and C; or by $-m = -2\cos\theta$, $\therefore\ \cos\theta = -.25$, giving L and M, as before.

Subtangents.—$S.\ T. = \dfrac{\rho^2 d\theta}{d\rho} = -\dfrac{(m+\cos\theta)^2}{(m^2-1)\sin\theta}$. When $\theta = 0°$ or $180°$ s. t. $= \infty$. When $\cos\theta = -m = -.5$, $\theta = 120°$ or $240°$; then s. t. $= 0$. This is at B, where the tangent and the radius vector coincide. The propriety of testing this will appear in *The quadrature of the area.*

The differential of the *area* of a polar curve is $\frac{1}{2}\rho^2 d\theta$;

$$\therefore\ dA = \frac{m^2 + 2m\cos\theta + \cos^2\theta}{2(m^2-1)^2}\,d\theta;$$

$$\therefore\ A = \frac{1}{2(m^2-1)^2}\left\{ m^2\theta + 2m\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right\} + C.$$

When $\theta = 0$, the area $= 0$, $\therefore\ C = 0$.

" $\theta = 120°$, $A = \frac{2}{9}(2\pi + 3\sin\theta) = 1.9736$.

" $\theta = 180°$, $A = \frac{2}{9}\,.\,3\pi = \frac{2}{3}\pi \quad = 2.0944$.

The first of these areas is $BAKFM$. The second is $BAKFMBGC$, i. e., one half of the whole area and one half of the inner area $BGCH$. Hence one half of the inner area is .1208. (The recognized notation for this is

$$\int_{\theta=120^\circ}^{\theta=180^\circ} \tfrac{1}{2}\rho^2 d\theta = 2.0944 - C,$$

in which case $C = 1.9736$; area $=.1208$.) The whole area then is 3.9472, and the inner area is .2416.

Cubature of Volume.—Let any radius vector, as BP, be revolved about the initial line AD, describing the surface of a cone. Then let another radius vector from B to a point on the curve indefinitely near P, describe the surface of another cone. The volume between these surfaces will have a thickness of zero at B, and of $\rho d\theta$ at P. The base $= 2\pi PQ\rho d\theta = 2\pi\rho^2 \sin\theta d\theta$, and the volume $= \frac{1}{3}\rho 2\pi\rho^2 \sin\theta d\theta = \frac{2}{3}\pi\rho^3 \sin\theta d\theta$. For all values of ρ from 0 to —2 its intrinsic sign is minus; and we have to integrate $-\frac{2}{3}\pi\rho^3 \sin\theta d\theta$,

$$= \int -\frac{2\pi}{3(m^2-1)^3}\Big(m^3 + 3m^2\cos\theta + 3m\cos^2\theta + \cos^3\theta\Big)\sin\theta d\theta$$

$$= \frac{2\pi}{3(m^2-1)^3}\Big(m^3\cos\theta + 3m^2\frac{\cos^2\theta}{2} + m\cos^3\theta + \frac{\cos^4\theta}{4}\Big) + C$$

$$= -\frac{128\pi}{81}\Big(\tfrac{1}{8}\cos\theta + \tfrac{3}{8}\cos^2\theta + \tfrac{1}{2}\cos^3\theta + \tfrac{1}{4}\cos^4\theta\Big) + C.$$

For $\theta = 90^\circ$ $\cos\theta = 0$. Take the volume here $= 0$ and we get $C = 0$. For $\theta = 0^\circ$, $\cos\theta = 1$; $\therefore$ $V = -\frac{128}{81}\pi(\frac{1}{8} + \frac{3}{8} + \frac{1}{2} + \frac{1}{4}) = -\frac{160}{81}\pi$. The minus sign indicates that this volume is to the left of a plane passing through EF perpendicular to AB.

For $\theta = 120^\circ$, $\cos\theta = -\frac{1}{2}$; $\therefore$ $V = -\frac{128}{81}\pi(-\frac{1}{16} + \frac{3}{32} - \frac{1}{16} + \frac{1}{64}) = \frac{2}{81}\pi$. The plus sign shows that this volume is on the right of the aforesaid plane. The arithmetical sum of these two volumes is the entire volume described by $AFMB$, and is $= 2\pi =$ vol. of a cylinder whose altitude, and the diameter of whose base, $= 2 = AB$. Like the cylinder, its volume $= \frac{3}{2}$ of a sphere whose diameter is AB.

Interior Volume.—By this I mean the vol. described by BGC. We integrate between the limits $\theta = 120^\circ$, $\cos\theta = -\frac{1}{2}$ and $\theta = 180^\circ$, $\cos\theta = -1$.

$$\therefore \int_{\cos = -\frac{1}{2}}^{\cos = -1} = 0 - \frac{2\pi}{81} = -\frac{2\pi}{81}.$$

Hence the interior vol. lies wholly on the left of EBF, and is equal to the button-shaped vol. described by BMF.

Moreover, each of these two volumes is equal to the vol. of a cone whose altitude is BC or BF, i. e., $\frac{2}{3}$, and the diameter of whose base is the same; and the sum of the two volumes is equal to that of a sphere whose diameter equals BC or BF; so that the three round bodies are severally represented.

Test.—The following test is subjoined partly to certify the reader of the accuracy of the work thus far, and partly to present a neat method of integration which chances to offer itself. Resume the formulæ

$$dV = -\tfrac{2}{3}\pi\rho^3\sin\theta d\theta, \qquad \rho = \frac{m+\cos\theta}{m^2-1}.$$

Differentiating, $\sin\theta d\theta = -(m^2-1)d\rho = \tfrac{3}{4}d\rho$;

$$\therefore dV = -\tfrac{1}{2}\pi\rho^3 d\rho; \therefore V = -\tfrac{1}{8}\pi\rho^4 + C.$$

Pursuing the same plan as before, we make $V=0$ for $\theta = 90^0$; $\therefore \rho = -\tfrac{2}{3}$. Hence the integral is $-\tfrac{2}{81}\pi + C$; $\therefore C = \tfrac{2}{81}\pi$.

For $\rho = -2$, $V = -2\pi + \tfrac{2}{81}\pi = -\tfrac{160}{81}\pi$. For $\theta = 120^0$, $\rho = 0$; $\therefore V = C = \tfrac{2}{81}\pi$. For $\theta = 180^0$, $\rho = +\tfrac{2}{3}$; $\therefore V = -\tfrac{2}{81}\pi + \tfrac{2}{81}\pi$. The former of these, $-\tfrac{2}{81}\pi$, is the interior volume, and the latter is the small exterior volume on the right of the plane passing through EBF.

The singular resemblance of this curve to a section of peach or plum inclusive of the seed suggests the name of Prunoid or Plum Curve.

NOTE.—Looking though Haddon's "Examples and Solutions in Diff. Calc." (Weale's Series), I find two curves which may be regarded as particular cases of the Prunoid.

One, which he calls the Trisectrix, has for its equation, $r = a(2\cos\theta \pm 1)$.

In the equation of the Prunoid, I originally introduced the constant a, but afterward for simplicity's sake treated it as unity. But as Haddon employs it, I restore this constant and write the equation of the Prunoid,

$$r = a\,\frac{\pm m + \cos\theta}{\pm m^2 \pm 1}.$$

Let $m = \tfrac{1}{2}$ and $a = \tfrac{3}{2}a'$; also let the sign of m^2 be minus, then

$$r = a'(2\cos\theta \pm 1).$$

Haddon's second curve has $r = a\cos\tfrac{1}{3}\theta$. Here $m = 0$, $\theta = \tfrac{1}{3}\theta'$; sign of 1 is plus; $\therefore$ $\qquad r = a\cos\tfrac{1}{3}\theta'$.

The former of these two curves seems, by a remarkable coincidence, identical with the Prunoid for $m = \tfrac{1}{2}$ as drawn and described in the foregoing article, except that the curve lies in an opposite direction. But even this discrepancy can be eliminated by writing, as Haddon suggests,

$$r = a(2\cos\theta - 1);\ \text{also make } a = \tfrac{2}{3}.$$

For instance, Haddon finds the distance $AB = 3a$. I get it $-2a$. He gives no account of the origin of the equation in the first case, but in the second the equation grows out of this problem:

"Two points start from the opposite extremities of the diameter of a circle, and move with uniform velocity in the same direction round the circumference; their velocities are in the ratio of 2 : 1. Determine the locus

of the bisection of the chords which join the position of the two points, and find the polar subtangent of the curve."

The curve lies in a contrary direction to the figure in this article, and the pole is at C. $CA = 2CB$; therefore the curve is not identical with ours.

The equation of the Cardioid is $r = a(1+\cos\theta)$. Write the Prunoid thus, $a\dfrac{m+\cos\theta}{m^2+1}$. Let $m = 1$ and $a = 2a'$; then $a\dfrac{m+\cos\theta}{m^2+1} = a'(1+\cos\theta)$. As its name indicates, the Cardioid is shaped like a heart.

NOTE BY E. L. DE FOREST.—SINCE the paper concluded at page 9 has been in the printer's hands, I have found that the theorems there given relative to the lever arm and the radius of gyration, are comprehended under two more general ones which may be briefly stated as follows.

If any number of polynomials are multiplied together, the lever arm of the coefficients in the product, about the first one as a fulcrum, is equal to the sum of the lever arms in all the factors; and the square of the radius of gyration in the product, about the centre of parallel forces as an axis, is equal to the sum of the squares of the radii of gyration in all the factors. Proof of these and some other properties must be reserved for a future article.

A correction should be made in the sentence at middle of page 8, which is so worded as to be not strictly true nor consistent with what had been said before.

The centre of parallel forces does not precisely "coincide" with the greatest coefficient unless the rank qm of that centre is a whole number. In all other cases, their positions differ by a fraction of the interval Δx. When this difference exists, the quadratic mean error ε is to be obtained by reckoning the distances of the terms from the centre of forces, and not from the greatest term. The difference vanishes when m is infinite and Δx becomes dx.

EXTENSION OF THF METHOD OF LEAST SQUARES TO ANY NUMBER OF VARIABLES.

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ANY point, line or surface is in its most probable position when the sum of the squares of the normals upon it, from the given points, is a minimum (ANALYST, Vol. IV, p. 184). That is, the most probable position